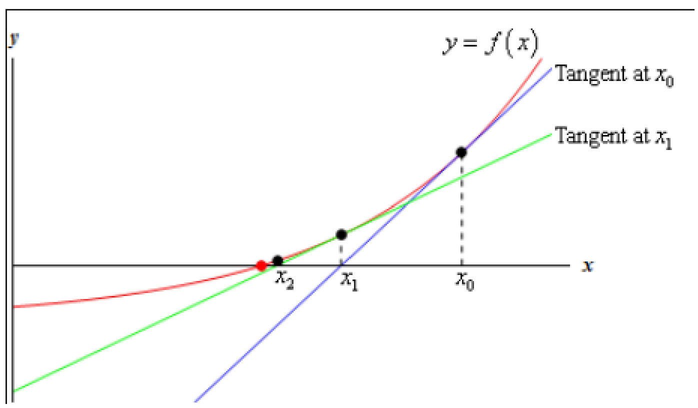
Review: Newton's Method - 11/4/16

1 Theory

Goal: Find the x value at which $f(x) = 0$.

We start with a guess: say we guess it hits the x axis at x_0. Our guess is a little off, so we want to try to make it closer. The way we do this is we take the tangent line to the curve. Remember, a tangent line is a straight line that approximates the curve. Since the tangent line is a straight line, we can easily find where it hits the x axis. Since the tangent line approximates the curve, the idea is that where the tangent line hits the x axis approximates where the curve hits the x axis. This point gives us a new approximation that is a little closer than our first guess. We keep on doing this, and each time we do, we end up closer to the right answer.



If we start with our original guess x_0, the first thing we need to do is find its tangent line. To do this, we need to know a point and the slope. Our point is $(x_0, f(x_0))$, and our slope is $f'(x_0)$, so the equation for our tangent line is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Now that we've found the tangent line, our next thing to do is find our next approximation, namely x_1. This approximation is where the tangent line hits the x axis, i.e. where $y = 0$, so we're looking for x_1 in the point $(x_1, 0)$. Plugging this into what we just found for the tangent line, we get

$$0 - f(x_0) = f'(x_0)(x_1 - x_0).$$

We're trying to find x_1, so we want to solve for x_1. We have $-f(x_0) = f'(x_0)x_1 - f'(x_0)x_0$, so $f'(x_0)x_0 - f(x_0) = f'(x_0)x_1$, so

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

In general, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2 Examples

Example 2.0.1 *Let* $x^3 - 2x - 5 = 0$*. We are going to approximate the root of this using* $x_0 = 2$ *as our first guess. Let* $f(x) = x^3 - 2x - 5$*, so* $f'(x) = 3x^2 - 2$*. Then* $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^3-2(2)-5}{3(2^2)-2} = 2 - \frac{(-1)}{10} = 2.1$*. To find* x_2*, we do* $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{(2.1)^3-2(2.1)-5}{3((2.1)^2)-2} \approx 2.0946$*. Let's do one more:* $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{(2.0946)^3-2(2.0946)-5}{3((2.0946)^2)-2} \approx 2.0946$*. Notice that we got the same answer here as we did last time. This means that no matter how many times we do this, we will still get this answer when we round to 4 decimal places. That means that we are correct up to four decimal places.*

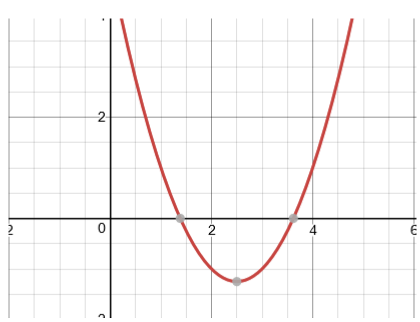
Example 2.0.2 *Find the* $\sqrt[6]{2}$ *correct to eight decimal places. In order to use Newton's method, we need a function, and we want the root of the function to be* $\sqrt[6]{2}$*. We are going to let* $f(x) = x^6 - 2$*. Notice that if we plug in* $\sqrt[5]{2}$ *into* $f(x)$*, we get 0. Then* $f'(x) = 6x^5$*. Let's start with* $x_0 = 1$*. We do this because* $1^6 = 1$*, but* $2^6 = 64$*. Thus* $\sqrt[6]{2}$ *is going to be somewhere in between 1 and 2, but 1 is going to be a lot closer to* $\sqrt[6]{2}$ *than 2 is. With* $x_0 = 1$*, we get* $x_1 = 1 - \frac{1^6-2}{6(1^5)} = 1 - \frac{(-1)}{6} \approx 1.16666667$*. Here I rounded it to 8 decimal places because that's how accurate we wanted our answer to be. I'm going to keep doing this until I get two* x_n *values that match. If I keep going, I get* $x_2 \approx 1.12644368$*,* $x_3 \approx 1.12249707$*,* $x_4 \approx 1.12246205$*,* $x_5 \approx 1.12246205$*. If we keep plugging this in, we will just keep getting that answer. Thus* $\sqrt[6]{2} \approx 1.12246205$*.*

3 Choosing a Starting Point

One way to choose a starting point is to use the Intermediate Value Theorem.

Example 3.0.3 *Find a starting value for Newton's method for solving the root of* $x^2 - 5x + 5 = f(x)$*. Let's try 0: If we plug in 0, we get 5. If we plug in 1, we get 1. If we plug in 2, we get* $4 - 10 + 5 = -1$*. By the IVT, since* $x^2 - 5x + 5$ *is continuous and* $f(2) < 0 < f(1)$*, then the value that we want is somewhere between 1 and 2. Since they are equally far away, either one is a good guess. We could also try 1.5, since it's half-way in between. It turns out that* $f(1.5) = -.25$*, which is even closer to 0, so we could also start with* $x_0 = 1.5$*. Note that we can start in different places and, as long as we can apply Newton's method there, we will still get the right answer. We want to pick a starting value that is closer because it will take less iterations that way.*

Another way to find a good starting point is by graphing the function. If we graph $x^2 - 5x + 5$, for example, we get



Looking at this graph, we see that the root that we're interested in is very close to 1.5, so this verifies that 1.5 is a good starting value.

3.1 Other Types of Examples

Newton's method works for most kinds of functions, not just polynomials.

Example 3.1.1 *We want to find the solution to* $\cos(x) = x$. *Remember, Newton's method helps us find where a function is zero, so we will define our function as* $f(x) = \cos(x) - x$. *Then* $f'(x) = -\sin(x) - 1$. *Then our general formula is* $x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1} = x_n + \frac{\cos(x_n) - x_n}{\sin(x_n) + 1}$. *If we graphed* $\cos(x) = x$, *we might guess that* $x_0 = .75$. *If we plug this in, we get* $x_1 = .75 + \frac{\cos(.75) - .75}{\sin(.75) + 1} \approx .7391$. *Then we can find* $x_2 = .7391 + \frac{\cos(.7391) - .7391}{\sin(.7391) + 1} \approx .7391$. *Thus this matches up to four decimal places, so this is the correct answer up to four decimal places.*

4 When does Newton's Method NOT Work?

There are a couple of situation in which Newton's method doesn't work.

1. There is no zero of the function (ex: $f(x) = e^x$)
2. $f'(x_n) = 0$ (ex: $f(x) = x^3 - 3x + 6$, then we can't choose $x_0 = 1$. In this case, we just choose a different starting value.)
3. Our guesses oscillate back and forth (ex: $f(x) = \sqrt[3]{x}$.)
4. $f'(x_n)$ is not defined (ex: $\sqrt[3]{x} = 0$ with $x_0 = 0$.)

Practice Problems

1. Let $x^4 - 4x + 1 = 0$. Find a solution correct up to 4 decimal places.

Solutions

1. Let $f(x) = x^4 - 4x + 1$, so $f'(x) = 4x^3 - 4$. We need to find x_0. Let's start with 0, then $f(0) = 1$. For 1, $f(1) = -2$. Since $f(x)$ is continuous and $f(1) < 0 < f(0)$, then there must be a root somewhere between 0 and 1. Since $f(0)$ is a little closer to 0 than $f(1)$ is, let's have $x_0 = 0$. Then $x_1 = 0 - \frac{0^4 - 4(0) + 1}{4(0)^3 - 4} = 0 - \frac{1}{-4} = .2500$. Here, we've rounded to 4 decimal places since that's how correct we want to be. If we keep going, we get $x_2 \approx .2510$, and $x_3 \approx .2510$. Since these are the same, we have found our answer correct up to four decimal places.